Assignment 13

This homework is due Monday May 4.

There are total 40 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 8.1–8.3 of Textbook.

RECALL that $C_\rho(z_0)$ denotes the circle of radius ρ centered at z_0 traversed counterclockwise.

(1) [10pt] Find residue at 0 of the following functions. (There are different ways to do that, some may be easier than others in each particular case.) (*Hint:* In (1f), remember that $e^{a+b} = e^a e^b$.)

(a) $z^{-1}e^z$.
(b) $\csc z$.
(c) $\frac{z^2+4z+5}{z^2+z}$.
(d) $z^{-3}\cos z$.
(e) $z^{-1}\sin z$.
(f) $e^{1+\frac{1}{z}}$.
(g) $\frac{e^{4z}-1}{\sin^2 z}$.
(h) $z^{-1}\csc z$.

(2) [8pt] Evaluate using residues:
(a) $\int_{C_1(1)} \frac{dz}{z^8-1}$.
(b) $\int_{C_2(0)} \frac{e^z\,dz}{z^3+z}$.
(c) $\int_{C_2(0)} \frac{\sin z\,dz}{4z^2-\pi^2}$.
(d) $\int_{C_1(0)} \frac{dz}{z^2\sin z}$.

(3) [5pt] Let f be analytic in a simply connected domain D, and let C be a positively oriented contour in D. If z_0 is the only zero of f in D and z_0 is inside C, then show that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = k$, where k is the order of the zero at z_0. (*Hint:* Write $f(z) = (z-z_0)^k g(z)$.)

(4) [5pt] Use residues to find the following trigonometric integrals. (Don't forget that in each case the answer is a *real* number.)
(a) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta$
(b) $\int_0^{2\pi} \frac{1}{(1+3\cos^2\theta)^2}d\theta$

(5) [12pt] Use residues to find the following improper integrals. (Don't forget that in each case the answer is a *real* number.)
(a) $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+16)^2}$.
(b) $\int_{-\infty}^{\infty} \frac{dx}{x^4+4}$.
(c) $\int_{-\infty}^{\infty} \frac{x^4dx}{x^6+1}$.
(d) $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+9)^2}$.
(e) $\int_{-\infty}^{\infty} \frac{x+3}{(x^2+9)^2}dx$.
(f) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, where $a>0$, $b>0$.
(g) $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+a^2)^3}$, where $a>0$.